Ex 92 p. 259 $\quad I = \int_0^1 \frac{x^2 e^x}{1+x}\,dx \qquad f(x) = \frac{e^x}{1+x}$ sur $[0,1]$

1°) $f$ est dérivable sur $[0,1]$ comme quotient de fonctions dérivables à dénominateur non nul.

$u(x) = e^x \qquad u'(x) = e^x$

$v(x) = 1+x \qquad v'(x) = 1$

$\forall x \in [0,1] \quad f'(x) = \frac{e^x(1+x) - e^x}{(1+x)^2} = \frac{xe^x}{(1+x)^2}$

| $x$ | $0$ | | $1$ |
|---|---|---|---|
| $x$ | $0$ | $+$ | |
| $e^x$ | | $+$ | |
| $(1+x)^2$ | | $+$ | |
| $f'(x)$ | $0$ | $+$ | |
| $f$ | $1$ | $\nearrow$ | $\frac{e}{2}$ |

2°) $S_n = \sum_{k=0}^{n} f\left(\frac{k}{5}\right) = f\left(\frac{0}{5}\right) + f\left(\frac{1}{5}\right) + \dots + f\left(\frac{n}{5}\right)$

a) $x \in \left[\frac{k}{5} ; \frac{k+1}{5}\right]$ $\qquad \int_a^b M\,dx = M(b-a)$

$\Leftrightarrow \frac{k}{5} \leq x \leq \frac{k+1}{5}$

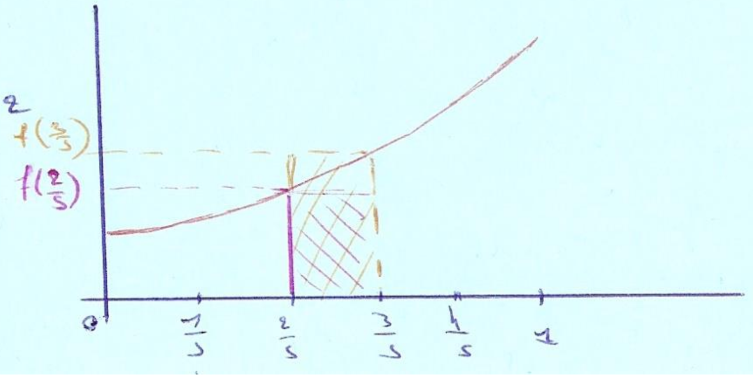
$\Leftrightarrow f\left(\frac{k}{5}\right) \leq f(x) \leq f\left(\frac{k+1}{5}\right)$ car $f$ est strict $\nearrow$ sur $\left[\frac{k}{5} ; \frac{k+1}{5}\right]$

$\Rightarrow \int_{\frac{k}{5}}^{\frac{k+1}{5}} f\left(\frac{k}{5}\right)dx \leq \int_{\frac{k}{5}}^{\frac{k+1}{5}} f(x)\,dx \leq \int_{\frac{k}{5}}^{\frac{k+1}{5}} f\left(\frac{k+1}{5}\right)dx$

$\Leftrightarrow f\left(\frac{k}{5}\right)\left(\frac{k+1}{5} - \frac{k}{5}\right) \leq \int_{\frac{k}{5}}^{\frac{k+1}{5}} f(x)\,dx \leq f\left(\frac{k+1}{5}\right)\left(\frac{k+1}{5} - \frac{k}{5}\right)$

$\Leftrightarrow \frac{1}{5} f\left(\frac{k}{5}\right) \leq \int_{\frac{k}{5}}^{\frac{k+1}{5}} f(x)\,dx \leq \frac{1}{5} f\left(\frac{k+1}{5}\right)$

pour $k = 2$



b) $k=0 \quad \frac{1}{5} f\left(\frac{0}{5}\right) \leq \int_0^{\frac{1}{5}} f(x)\,dx \leq \frac{1}{5} f\left(\frac{1}{5}\right)$

$k=1 \quad \frac{1}{5} f\left(\frac{1}{5}\right) \leq \int_{\frac{1}{5}}^{\frac{2}{5}} f(x)\,dx \leq \frac{1}{5} f\left(\frac{2}{5}\right)$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$k=4 \quad \frac{1}{5} f\left(\frac{4}{5}\right) \leq \int_{\frac{4}{5}}^{\frac{5}{5}} f(x)\,dx \leq \frac{1}{5} f\left(\frac{5}{5}\right)$

somme membre à membre

$$\frac{1}{5} f(0) + \frac{1}{5} f\left(\frac{1}{5}\right) + \dots + \frac{1}{5} f\left(\frac{4}{5}\right) \leq \int_0^1 f(x)\,dx \leq \frac{1}{5} f\left(\frac{1}{5}\right) + \frac{1}{5} f\left(\frac{2}{5}\right) + \dots + \frac{1}{5} f(1)$$

$$\Leftrightarrow \frac{1}{5}\left(f(0) + \dots + f\left(\frac{4}{5}\right)\right) \leq \int_0^1 f(x)\,dx \leq \frac{1}{5}\left(f\left(\frac{1}{5}\right) + \dots + f(1)\right)$$

$$\Leftrightarrow \frac{1}{5} S_4 \leq \int_0^1 f(x)\,dx \leq \frac{1}{5}\left(S_5 - f(0)\right)$$

$$\Leftrightarrow \frac{1}{5} S_4 \leq \int_0^1 f(x)\,dx \leq \frac{1}{5}\left(S_5 - 1\right).$$

c) A la calculatrice : on remarque que

$$S_{n+1} = \underbrace{f\left(\frac{0}{5}\right) + f\left(\frac{1}{5}\right) + \dots \dots + f\left(\frac{n}{5}\right)} + f\left(\frac{n+1}{5}\right)$$

$$\Leftrightarrow S_{n+1} = S_n + f\left(\frac{n+1}{5}\right)$$

formule de récurrence de $S_n$

on obtient $S_4 \simeq 5,458660579$

$S_5 \simeq 6,817801492$

On en déduit que $1,09173 \leq \int_0^1 \frac{e^x}{1+x}\,dx \leq 1,1636$

3°) a) $\forall x \in [0,1] \quad 1 - x + \frac{x^2}{1+x}$

$$= \frac{(1-x)(1+x) + x^2}{1+x} = \frac{1 - x^2 + x^2}{1+x} = \frac{1}{1+x}$$

b) $\int_0^1 \frac{e^x}{1+x}\,dx = \int_0^1 e^x \times \frac{1}{1+x}\,dx.$

$= \int_0^1 \left(1-x+\frac{x^2}{1+x}\right)e^x\,dx$

$= \int_0^1 (1-x)e^x\,dx + \int_0^1 \frac{x^2}{1+x}e^x\,dx$

$\Leftrightarrow \int_0^1 \frac{e^x}{1+x}\,dx = \int_0^1 (1-x)e^x\,dx + I.$

c) $H(x) = (ax+b)e^x$ est une primitive de $h(x) = (1-x)e^x$

$\Leftrightarrow H'(x) = h(x)$

$u(x) = ax+b \qquad u'(x) = a$

$v(x) = e^x \qquad v'(x) = e^x$

$H'(x) = ae^x + (ax+b)e^x = (ax+b+a)e^x$

par identification des coefficients

$\begin{cases} a = -1 \\ b+a = 1 \end{cases} \Leftrightarrow \begin{cases} a = -1 \\ b = 1-a = 2 \end{cases}$

<u>Cl</u>: $H(x) = (-x+2)e^x$

d) D'après 3°) b)

$I = \int_0^1 \frac{e^x}{1+x}\,dx - \int_0^1 (1-x)e^x\,dx$

$= \int_0^1 \frac{e^x}{1+x}\,dx - \left(H(1) - H(0)\right)$